Tensor network wavefunction methods

• Old wisdom: wavefunction too complicated - ED, even with Lanczos/targeted methods scales exponentially
  → map to few/single particle problem: DFT, mean-field
  → parameterize w/ small number of parameters
    variational method - not systematic/unbiased or controlled
  → map to "classical" problem & sample (QMC of path integral)

• Recent wisdom (quantum info.):
  → entanglement: amount of info. needed about subsystem B
to know state of subsystem A

entangled = complicated / less local

But we are discovering that "physical" states esp. ground states
are only "lightly" entangled.
Leading to new tools for storing, manipulating,
optimizing wavefunctions.

Rather complete knowledge in 1d

• Matrix product states
  Efficient, convenient way of representing all 1d wavefunctions
  having a correlation length
  Can represent gapless (power-law correlated) states controllably, too
  Only bias is low-entanglement (very weak for ground states)

Comes with natural optimization algorithm: DMRG

Setting: 1d lattice models

**Ex #1:** Heisenberg spin chain $s = 1/2$. $H = \sum_j \vec{S}_j \cdot \vec{S}_{j+1}$

Local Hilbert space is single spin $1/2$ $|\sigma\rangle \in \{ |\uparrow\rangle, |\downarrow\rangle \}$

General wavefunction of the form

$$|\psi\rangle = \sum_{s_1, s_2, \ldots, s_N} \psi^{s_1, s_2, \ldots, s_N} |s_1\rangle \otimes |s_2\rangle \otimes |s_3\rangle \cdots \otimes |s_N\rangle$$

On two sites ($N=2$), ground state is singlet

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} |\uparrow\uparrow\rangle - \frac{1}{\sqrt{2}} |\downarrow\downarrow\rangle$$

$$\psi^{s_1, s_2} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

For $N>2$, more complicated singlet with quasi-long range (power law) antiferromagnetic correlations

**Ex #2:** 1d "electronic structure"/electrons in continuum

Can discretize continuum using grid basis

$$\psi(x_0, x_1, \ldots, x_N) \rightarrow |\psi\rangle = \sum_{s_1, s_2, \ldots, s_N} \psi(s_1, s_2, s_3, \ldots, s_N) |s_1\rangle \otimes |s_2\rangle \otimes \cdots \otimes |s_N\rangle$$

$s_j \in \{ 0, 1, \uparrow, \downarrow \}$

Like binning real space

$$\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\uparrow & 0 & 0 & \uparrow & 0 & 0 \\
\downarrow & 0 & 0 & \downarrow & 0 & 0 \\
\end{array} \rightarrow \text{amplitude } \psi(0, 0, 0, 0, 0, 0)$$

$$H = -\frac{1}{2a} \int \psi^\dagger(x) \vec{\delta} \cdot \vec{\delta} \psi(x) \rightarrow H_{\text{grid}} = \frac{-1}{2a^2} \sum_{j_0} (c_{j_0}^+ c_{j_0} - 2 c_{j_0}^+ c_{j_0} + c_{j_0}^+ c_{j_0})$$
Getting the ground state wavefunction:

1. Try putting Hamiltonian as a matrix into a full diagonalization code

\[ H = \begin{pmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{pmatrix} \rightarrow U E U^T \quad \text{cost: } (d^N)^3 = d^{3N}. \]

2. "Target" low-energy states - Lanczos, Davidson
work involved is multiplying \( H \{ \psi_n \} = \{ \psi_{n+1} \} \)
\( \{ \psi_1, \psi_2, \ldots, \psi_n \} \)
\[ m \text{ of the order of } 10^2 \sim 1000 \]
Use sparsity of \( H \), \( H^2 \psi \) cost \( \propto d^3 N \), still \( \sim d^N \)
Find ground state, low-lying excited states in \( H \{ \psi_n \} \) basis.

3. DMRG (White)
Problem with ED, even with targeting, is exponentially large Hilbert space basis
Can one identify a smaller basis sufficient for targeting low-energy states?
The "job" of the ground state is to minimize the energy; Hamiltonian is sum of local operators
Which part of the ground state wavefunction do local operators (e.g. terms in \( H \)) see?
Consider expectation value of operator $\hat{n}_2$ (for concreteness) measuring density on site 2.

$$\langle \Psi | \hat{n}_2 | \Psi \rangle = \sum_{s_1, s_2, s_3 \cdots s_N} \overline{\Psi}_{s_1, s_2, s_3 \cdots s_N} n_{s_2} \psi_{s_1, s_2, s_3 \cdots s_N}$$

$$= \sum_{s_1, s_2, s_3} n_{s_2} \rho_{s_1, s_2}$$

where

$$\rho_{s_1, s_2} = \sum_{s_3, s_4 \cdots s_N} \overline{\Psi}_{s_1, s_2, s_3, s_4 \cdots s_N} \psi_{s_1, s_2, s_3, s_4 \cdots s_N}$$

$\rho$ is called a reduced density matrix.

Only information about $|\Psi\rangle$ we need to understand properties on sites 1 & 2.

Reduced density matrices Hermitian & positive semi-definite.

Can diagonalize as $\rho_{s_1, s_2} = U_{s_1, s_2}^\dagger \rho_{s_1, s_2}^\alpha \ U_{s_1, s_2}^{s_1, s_2}$

$$\rho_{s_1, s_2}^\alpha = \begin{bmatrix} \rho_1 & \rho_2 & \cdots \end{bmatrix}; \sum \rho_j = 1; \quad U = \begin{bmatrix} U_1 & U_2 & \cdots \end{bmatrix}$$

Can interpret columns of $U$ as wavefunctions for the $s_1, s_2$ subsystem $|1\alpha\rangle = U_{s_1, s_2}^\dagger |s_1, s_2\rangle$ and $\rho_\alpha$ as probabilities for finding these sites in the state $|1\alpha\rangle$. 
In the density-matrix eigenbasis ("Schmidt basis") expectation values of local operators take the form

\[ \langle \psi | \hat{n}_z | \psi \rangle = \sum_{\alpha} \langle \alpha | \hat{n}_z | \alpha \rangle \rho_\alpha \]

For gapped 1d systems, the \( \rho_\alpha \) fall very fast (the number of large \( \rho_\alpha \) is independent of system size) (Hastings)

\[ \text{(schematic)} \]

\[ \frac{\epsilon}{\epsilon} \]

\[ 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \alpha \]

Suggests an approximation: truncate (project out) states \( \alpha \) having \( \rho_\alpha \) less than some cutoff

Guaranteed to preserve all local properties up to an error proportional to the sum of discarded probabilities.

Origin of the term "density matrix renormalization group" practitioners often talk about the "number of states left" in DMRG - refers to keeping first \( m \) states, discarding (projecting out) the rest.

Basic strategy: apply such an approximation for all bipartitions of the system, not just sites...

Result will be a matrix product state.

(From this point of view, wavefunction is a chain of projectors.)

Gives highly efficient (optimal in some cases) compression scheme for 1d many-body states.
Density matrix examples

Ex #1: singlet

\[ \psi_{1, s_2} = \begin{bmatrix} \uparrow \downarrow \end{bmatrix} \rightarrow \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & 0 \end{bmatrix} \]

Ex #2:

\[ \psi_{1, s_2} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \frac{1}{2} \left( \uparrow \uparrow + \uparrow \downarrow + \downarrow \uparrow + \downarrow \downarrow \right) \]

\[ e^{s_1} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \]

To carry out the compression, first truncate \( U \):

\[ U = \begin{bmatrix} u_{1, s_2} & u_{2, s_2} & u_{3, s_2} & u_{4, s_2} \\ & & & \end{bmatrix} \rightarrow \begin{bmatrix} u_1 & u_2 \end{bmatrix} \]

Rewrite wavefunction in (truncated) Schmidt basis:

\[ |\psi\rangle = \sum_{s_1 \cdots s_n} \psi_{s_1, s_2, s_3 \cdots s_n} |s_1, s_2, s_3 \cdots s_n\rangle = \sum_{\alpha s_1 \cdots s_n} \psi_{\alpha s_1 \cdots s_n} |\alpha_{s_1} s_2, s_3 \cdots s_n\rangle \]

\[ \psi_{\alpha s_1 \cdots s_n} = \sum_{s_1 s_2} \psi_{s_1 s_2 s_3 \cdots s_n} U_{s_1 s_2}^{\alpha s_2} \]

For later purpose of optimizing wavefunction to search for ground state, transform Hamiltonian terms into this basis too:

\[ \langle s_1, s_2 | \hat{h}_{12} | s_1', s_2' \rangle \rightarrow \langle \alpha_{s_1} \hat{h}_{12} | \alpha_{s_2} \rangle = \sum_{s_1 s_2} U_{s_1 s_2}^{\alpha s_2} \langle s_1 s_2 | \hat{h}_{12} | s_1' s_2' \rangle U_{s_1 s_2}^{\alpha s_2} \]

save in memory for later
At this point, useful to introduce diagrammatic notation for tensors

A rank-0 tensor is a blob with lines emanating from it.

E.g., vector - rank 1 $v^i = \delta^i_0$

matrix - rank 2 $M^{ij} = \delta^{ij}$

many-body wavefunction - rank $N$

$\psi = \psi_1 \ldots \psi_N$

Similar to Einstein summation, connecting two index lines implies the indices are summed or contracted.

E.g., matrix-vector multiplication

$$\sum_j M^{ij} v_j \leftrightarrow \quad 0_j^0$$

Trace of product of matrices

$$\text{Tr}[AB] = \sum_{ij} A^i_j B^j_i \leftrightarrow \quad 0_0$$

Can observe rank of the result from number of open lines remaining.
Now let's use this notation to picture the procedure of projecting out the unimportant states of the density matrix of the first two sites.

Recall \( |\psi\rangle = \sum_{S_1S_2\ldots S_N} \chi_{S_1S_2\ldots S_N} |S_1S_2\rangle |S_3\ldots S_N\rangle \approx \sum_{S_2S_3\ldots S_N} \phi_{S_2S_3\ldots S_N}^0 |S_2\rangle |S_3\ldots S_N\rangle \)

where \( \phi_{S_2S_3\ldots S_N}^0 = \sum_{S_1S_2} \psi_{S_1S_2S_3\ldots S_N} U_{S_1S_2}^+ \phi_{S_2S_3\ldots S_N} \)

\[ \Rightarrow \psi_{S_1S_2S_3\ldots S_N} \approx \sum_{\alpha_2} U_{\alpha_2}^+ \psi_{S_1S_2S_3\ldots S_N} \]

MPS idea is to repeat this iteratively thinking of \( \alpha_2 \) as a "site"

\[ \Rightarrow \psi_{S_1S_2S_3\ldots S_N} \approx \sum_{\alpha_2} U_{\alpha_2}^+ \psi_{S_1S_2S_3\ldots S_N} \]

Computed \( U_3 \) by diagonalizing density matrix

\[ P_{\alpha_2S_3} = \sum_{S_4S_5\ldots S_N} \psi_{\alpha_2S_3S_4S_5\ldots S_N}^* \psi_{\alpha_2S_3S_4S_5\ldots S_N} \]

\[ = \sum_{\alpha_2} U_{\alpha_2}^+ P_{\alpha_2} U_{\alpha_2}^+ \]

\[ \Rightarrow U_{\alpha_2}^+ P_{\alpha_2} U_{\alpha_2}^+ = \]
At this point it's useful to observe we could have obtained these decompositions using the singular value decomposition (SVD) instead.

Recall SVD algorithm computes for any possibly rectangular matrix

\[ M_{ij} = U \alpha' D_{\alpha\alpha'} V^\dagger_{\alpha' j} \]

such that

\[ D_{\alpha\alpha'} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ \vdots & \vdots \\ 0 & \lambda_n \end{bmatrix} \]

\( \lambda_k \)'s real, \( \geq 0 \), decr. order

and \( U^\dagger U = I \Rightarrow \sum \lambda_k U_{\alpha'i} U^*_{\alpha'j} = \delta_{\alpha'i} \delta_{\alpha'j} \) \( (U U^\dagger \neq I!) \)

\[ V V^\dagger = I \Rightarrow \sum \lambda_k V^*_{\alpha'j} V_{\alpha'i} = \delta_{\alpha'i} \delta_{\alpha'j} \] \( (V V^\dagger \neq I!) \)

Diagrammatically, \( M = U \alpha D \alpha V^\dagger \)

and \( (U^\dagger V^\dagger = \begin{pmatrix} \frac{1}{\sqrt{n}} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \frac{1}{\sqrt{n}} \end{pmatrix} \)

Importantly, provides controlled approx for \( M \)

Define \( \tilde{M}_n \) as product \( U \tilde{D}_n V \) where

\[ \tilde{D}_n = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \]

then \( \| \tilde{M}_n - M \|^2 = \sum_{\alpha = n+1}^{n_{\text{max}}} \lambda_\alpha^2 \) \( \leq \text{"discarded weight"} \)
Returning to many-body wavefunction, think of as (highly rectangular) matrix

\[ \psi(s_1 s_2 s_3 s_4 \ldots s_N) = U \alpha x^2 D x^2 V \]

SVD

\[ \psi(s_1 s_2) = U \alpha x^2 D x^2 V \]

Can show that \( U \) is same as obtained from density matrix \( \rho_{s_1 s_2} = \)

\[ \psi(s_1 s_2 \ldots s_N) = U \alpha x^2 D x^2 V \]

Multiplying \( D \) into \( V \) get same decomposition as before

\[ \psi(s_1 s_2 \ldots s_N) = U \alpha x^2 D x^2 V \]

Armed w/ simpler SVD approach, apply recursively

\[ \psi(s_1 s_2 s_3 s_4 \ldots s_N) = U_2 \alpha_2 D_2 \alpha_2 V_2 \]

etc.

\[ U_1 U_2 U_3 U_4 U_5 U_6 \]

matrix product state in "right canonical form"

What have we accomplished?
Say when doing SVD's we truncate to at most \( m \) singular values at each step.

Obtain variational \( \psi \) ansatz/description of particular \( \psi \) using only \( \sim N m^2 \) parameters. Compare to \( d^N \) for ED.

Of course, only useful if accurate for some moderate sized \( m \).

Hastings has shown [J Stat Mech P08024 (2007)] there exists for gapped, finite range 1d Hamiltonians, some \( m \) independent of system size such that ground state well approximated by an MPS of "bond dimension" \( m \)

MPS also extremely efficient for computing properties of \( \psi \) (or operations on \( \psi \))

Say we want expectation value of some product of local operators e.g. a correlator:

\[
\begin{array}{cccccc}
    & & & & & \\
\| & & & & & \\
\| & & & & & \\
\| & & & & & \\
\| & & & & & \\
\| & & & & & \\
\| & & & & & \\
\| & & & & & \\
\| & & & & & \\
\| & & & & & \\
\| & & & & & \\
\| & & & & & \\
\end{array}
\]

By contracting piece-by-piece can compute with \( \sim N m^3 \) cost.

Again, compare to \( d^N \) for ED.
Can do even better for single local operator. Consider $\langle \psi | \hat{O} | \psi \rangle$ acting on site $N$.

\[
\begin{bmatrix}
I & I & \cdots & I & 0 \\
0 & u_1 & u_2 & \cdots & u_N
\end{bmatrix}
\]

All $u_j$ have property \[
\begin{bmatrix}
0 \\
\vdots \\
0 \\
u_j^T
\end{bmatrix} = C \begin{bmatrix}
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
\]

\[
\Rightarrow \langle \psi | \hat{O} | \psi \rangle = \sum_j C u_j^T
\]

Only $m$ cost! No factor of $N$.

We have exploited MPS gauge:

Many possible gauges $\begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}$

Handful of important useful ones. Shift gauge by contracting pairs of site tensors and computing SVD:

\[
\begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]
Can go from \[ u u u u u \]
to \[ u u u \frac{i}{\hbar} v v \ldots \frac{i}{\hbar} v v \] "mixed canonical"

Site \( j \) is said to be the "orthogonality center" (or because all its indices label orthonormal states

\[ |\psi\rangle = \sum \psi_{\alpha j} \alpha_j |\alpha_j\rangle \] \[ |\alpha_j\rangle \]

\[ \langle \alpha_j' | \alpha_j \rangle = \prod \frac{T_{\alpha_j}}{u_j} = C_{\alpha_j' \alpha_j} = \delta_{\alpha_j' \alpha_j}, \text{ similarly for } \langle \alpha_j' | \alpha_j \rangle \]

Measuring expectation value \[ \langle \psi | \alpha_j' \psi \rangle \]

Intuitively, what are the \( |\alpha_j\rangle \) states? (Called "Schmidt states"; density matrix eigenstates.) They are precisely the states needed to describe the many-body w.f.

For a ground state of a gapped Hamiltonian with correlation length \( \xi \), can show that \( |\alpha_j\rangle \) differ from ground state only within \( \xi \) of the cut

\[ \langle \alpha_j | \alpha_{j'} \rangle \uparrow \xi \]

- \[ \langle \alpha_j | \alpha_{j'} \rangle \]

\[ \alpha_{j'} \]
Entanglement entropy

Defined as $S = -\text{Tr}[\rho_A \ln \rho_A]$

where $\rho_A$ is the reduced density matrix describing some subsystem $A$.

In terms of eigenvalues $\lambda \rho_A^3$ of $\rho_A$, $S = -\sum \lambda \ln \lambda$

Ex 1 singlet $|s_{s_2}^1\rangle = \begin{bmatrix} 0 & \sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{2}} & 0 \end{bmatrix} \Rightarrow \rho_{s_i}^1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \sqrt{2} |s = \ln 2$

Ex 2 $1 \leftrightarrow 1 \rightarrow$ $|s_{s_2}^1\rangle = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \rho_{s_i}^1 = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{16} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow S = 0$

Measures degree of correlation (statistical) between systems $A \& B$.

For an MPS in mixed-canonical gauge, easily access entropy of left $(1, 2, \ldots, s)$ with right $(i+1, \ldots, N)$

\[
\begin{array}{ccc}
\psi_1 & \psi_2 & \psi_3 \\
\psi_{s+1} & | & \psi_{s+1} \\
\psi_{s+1} & | & \psi_{s+1} \\
\end{array}
\]

Can show eigenvalues of $\rho_L$ (= evals of $\rho_{AB}$) are squares of singular values $\lambda_x = \rho_{ij}$ from diagonal of $D_{ij}$.

If bond dimension of MPS is $m$, this implies a maximum entropy $\log(m)$.

MPS naturally obey the boundary law ("area law")
Before discussing DMRG algorithm, useful to construct matrix product operator (MPO) for the Hamiltonian.

Tensor network which mimics MPS but for operators

\[ s_i s_2 s_3 s_4 s_5 \cdots s_n \]

Can view each tensor as matrix of operators

\[
\begin{align*}
\hat{t}^{i+1} & = \hat{s}_i \hat{t}^i \hat{s}_i^{-1} \\
\hat{s}_i & = \begin{pmatrix} \hat{e}_{1i} & \cdots & \hat{e}_{ni} \end{pmatrix}
\end{align*}
\]

Simple to construct sums of finite-range operators in 1D

\[
\begin{pmatrix} o \end{pmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\
0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\
0 & I \end{bmatrix} \cdots \begin{bmatrix} I & 0 \\
0 & I \end{bmatrix} \begin{bmatrix} 0 \\
1 \end{bmatrix}
\]

\[
\begin{align*}
o & = \sum_j \hat{o}_j^2 \\
\begin{bmatrix} 0 & 1 \\
1 & 0 \end{bmatrix} & = \begin{bmatrix} I & 0 \\
0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\
0 & I \end{bmatrix} \cdots \begin{bmatrix} I & 0 \\
0 & I \end{bmatrix} \begin{bmatrix} 0 \\
1 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix} 1 & 0 \\
0 & I \end{bmatrix} & \begin{bmatrix} I & 0 \\
0 & I \end{bmatrix} \cdots \begin{bmatrix} I & 0 \\
0 & I \end{bmatrix} \begin{bmatrix} 0 \\
1 \end{bmatrix} & = \sum_j \hat{o}_j^2 \hat{o}_{j+1}^2 - h \sum_j \hat{o}_j^z
\end{align*}
\]
MPO makes it convenient & efficient to compute e.g. total energy

Also convenient for formulating DMRG algorithm.

Recall if OCC at site \( j \),

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
\nu_1 & \nu_2 & \nu_3 & \nu_4 \\
\end{pmatrix}
\]

So \( \nu_j \) changes from lattice basis \( u_1, u_2, \ldots, u_{\nu_j} \) to \( b_j \) basis & projects into first \( m \) \( \alpha_j \)'s

Hamiltonian in \( \{ \phi_{j} \}, \{ \phi_{j+1} \}, \{ \phi_{j+2} \} \) basis:

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

Idea of ground-state DMRG is to compute ground state of \( \hat{H} \) at each bond, use to update wavefunction.

Importantly, if MPS already approximates ground state well, is fixed point of this procedure.

If not, hope is that updating w.r.t. this way enriches matrix product basis i.e. obtain better set of states \( \{ \nu_j \} \).
Steps of DMRG

1. Given "projected Hamiltonian" \( \hat{H} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \)
   and initial wavefunction \( \psi \)
   find ground state (approximately) of \( \hat{H} \)
   using iterative exact diagonalization code (Chan, Davidson).

2. SVD new approx ground state to restore MPS form
   \( \psi_{\text{SVD}} = \begin{pmatrix} u & v \end{pmatrix} D \begin{pmatrix} u \\ v \end{pmatrix} \)
   Truncate to only \( m \) states (in singular values)
   to control cost of calculation
   (This is the reason to expose two sites.)

3. If next two sites are to the right, say,
   contract (multiply) \( D \) with \( V \) to shift gauge
   to the right,
   \( \begin{pmatrix} u & v \end{pmatrix} D V \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \)

4. Build \( \hat{H}_{ij} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \)
   (or pieces of it) by "growing"
   left edge tensor
   (if exposing next two sites to the right)
   \( \hat{H} \)
   Recover previous right edge
   tensor saved in memory

* Diagrammatically, \( \psi_{\phi} \) looks like

\[ \hat{H} \begin{pmatrix} \phi \\ \phi \end{pmatrix} = \begin{pmatrix} \phi \\ \phi \end{pmatrix} \]
Leads to "sweeping" algorithm

\[
\begin{align*}
&\vdots \\
&\vdots \\
&\vdots \\
&\vdots
\end{align*}
\]

In 1d, short-range lattice made convergence is exponentially fast in number of sweeps over system.

Other topics:

- MP
ds for long-range
- Collapsing on MPS: perfect sampling (4 METIS)
- Using DMRG for quasi-2d